Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-10

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. In the formula

$$(*)\qquad \iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}u\,\mathrm{d}v$$

the region S is the 2D surface in $\mathbb{R}^3$ to be integrated over, f is the function to be integrated, $\mathbf{r} = \langle x, y, z\rangle$ is expressed as a function of u, v via a *parametrization* of the surface. The region D is the 2D region in the uv-plane that sweeps out the surface S with the given parametrization.

(a) In the formula

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_D f(T(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v$$

we want to integrate f over some region R in the xy-plane, and T is a (re)parametrization of that region in terms of u, v. D is the corresponding region in the uv-plane.

We can interpret R as a surface contained entirely in the $z = 0$ plane, i.e. a surface in $\mathbb{R}^3$ with the parametrization $(x,y) = T(u,v)$ and $z = 0$. In this situation,

$$|\mathbf{r}_u \times \mathbf{r}_v| = \left|\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{bmatrix}\right| = \left|\left\langle 0, 0, \frac{\partial(x,y)}{\partial(u,v)}\right\rangle\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

so we recover the absolute value of the Jacobian determinant. Note that with the exception of the vertical bars on the very last term, which really do mean "absolute value," all the preceding bars mean "vector magnitude."

(b) In the formula

$$\iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\,\mathrm{d}x\,\mathrm{d}y$$

the function f does not play the role that it does in the preceding formulas! In this situation, the function being integrated is just 1, while $z = f(x,y)$ is a description of the surface being integrated over. D is the projection of the surface into the xy-plane, or in other words, the corresponding region of the xy-plane if we take the evident parametrization $\mathbf{r} = (x, y, f(x,y))$ in terms of x, y. In this situation,

$$|\mathbf{r}_u \times \mathbf{r}_v| = \left|\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{bmatrix}\right| = |\langle -f_x, -f_y, 1\rangle| = \sqrt{(f_x)^2 + (f_y)^2 + 1}.$$

Again, note that all the vertical bars mean "vector magnitude" here.

Answers to computations

Problem 1.

(a) The part with $1 \le x \le 2$ can be described with the equivalent equation $x = \sqrt{y^2+z^2}$, so one possible parametrization is

$$x = \sqrt{y^2+z^2}$$
$$y = y$$
$$z = z$$

where the parameters are y, z. The corresponding region in the yz-plane is the annulus $1 \le \sqrt{y^2+z^2} \le 2$. If you were doing an integral though, you'd probably reparametrize at this point in terms of polar, letting $y = u\cos v$ and

$z = u \sin v$, resulting in the parametrization

$$x = u$$
$$y = u \cos v$$
$$z = u \sin v$$

and the region in the uv-plane is $1 \le u \le 2$ and $0 \le v \le 2\pi$.

(b) We know how to use spherical coordinates to reparametrize $\mathbb{R}^3$ with three parameters ρ, ϕ, θ. Note that the sphere is $\rho = \sqrt{2}$, so the following will do:

$$x = \sqrt{2} \sin \phi \cos \theta$$
$$y = \sqrt{2} \sin \phi \sin \theta$$
$$z = \sqrt{2} \cos \phi$$

with parameter bounds $0 \le \theta \le 2\pi$ and $0 \le \phi \le 3\pi/4$. This last bound comes from observing that $z \ge -1$ is $\sqrt{2}\cos\phi \ge -1$.

(c) The equation given is y in terms of x, z (it just happens to not depend on x). So we can just take

$$x = x$$
$$y = 1 - z^2$$
$$z = z$$

and the region in the xz-plane is $-3 \le x \le 3$, $-1 \le z \le 1$.

(d) I talked about how to parametrize surfaces of revolution like this in class. First we parametrize the curve to be rotated with one parameter, say u. We can achieve this with $x = 2 + \cos u$, $y = \sin u$.

Now, if we rotate a point (a, b) in the xy-plane about the y-axis, we end up with a circle of radius a in the plane $y = b$, and we parametrize it with another parameter, such as v. We get by $x = a \cos v, y = b, z = a \sin v$.

Finally we let (a, b) vary along the original parametrized curve—which is to say, we substitute $a = 2+\cos u, b = \sin u$ into the preceding and obtain the parametrization

$$x = (2 + \cos u) \cos v$$
$$y = \sin u$$
$$z = (2 + \cos u) \sin v$$

where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$.

(e) Since this is a graph like (a), we can again take the evident parametrization, which is to say

$$x = x$$
$$y = y$$
$$z = \arctan(y/x)$$

and the region in the xy-plane is just as described: $y \ge x/\sqrt{3}, y \le x, x^2 + y^2 \le$.

But again, if we were actually doing this as part of a calculus problem, we'd want to reparametrize—likely using polar coordinates. So let $x = u \cos v$, $y = u \sin v$ as usual:

$$x = u \cos v$$
$$y = u \sin v$$
$$z = v$$

and now the region in the uv-plane is described by $0 \le u \le 2$ and $\pi/6 \le v \le \pi/4$.

(f) First of all, how does one parametrize the line segment with endpoints $(3, 2, 3)$ and $(4, 5, 6)$? We take one point as the starting point, say $(3, 2, 3)$, and compute the vector to the other point: $\langle 1, 3, 3\rangle$. The parametrization of the line segment is then $\mathbf{r} = \langle 3, 2, 3\rangle + t\langle 1, 3, 3\rangle$ with t bounds $0 \le t \le 1$, since $t = 0$ gives the point $(3, 2, 3)$ and $t = 1$ gives the point $(4, 5, 6)$.

To parametrize the triangle with vertices $(3, 2, 3), (4, 5, 6), (4, 6, 5)$, we do something similar. We take one point, e.g. $(3, 2, 3)$, as the "starting point," which will correspond to $(u, v) = (0, 0)$. The vectors connecting it to the other two points are $\langle 1,3,3\rangle$ and $\langle 1,4,2\rangle$. Now we take the parametrization $\mathbf{r} = \langle 3,2,3\rangle + u\langle 1,3,3\rangle + v\langle 1,4,2\rangle$, which is to say

$$\begin{aligned} x &= 3 + u + v \\ y &= 2 + 3u + 4v \\ z &= 3 + 3u + 2v. \end{aligned}$$

Without bounds on u, v, this gives a parametrization of the plane that passes through the three given points. If we just want the triangle though, notice that the corners of the triangle correspond to $(u, v) = (0,0), (1,0), (0,1)$. Since this parametrization is *linear*, lines correspond to lines (this argument only works because the parametrization is linear!). So the corresponding region in the uv-plane is just the triangle with those three vertices.